Fundamental Algorithms

Chapter 7: Parallel Sorting

Jan Křetínský

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Sequential MergeSort

```
MergeSort(A:Array[1..n]) {
  if n > 1 then {
    m := floor(n/2);
    create array L[1...m];
    for i from 1 to m do { L[i] := A[i]; }

    create array R[1...n-m];
    for i from 1 to n-m do { R[i] := A[m+i]; }

    MergeSort(L);
    MergeSort(R);

    Merge(L,R,A);
  }
}
```

(How) can we parallelise MergeSort?

MergeSort in Parallel?

```
MergeSortPar(A:Array[1..n]) {
  if n > 1 then {
    m := floor(n/2);

    do in parallel {
      create array L[1...m];
      for i from 1 to m do { L[i] := A[i]; }
      MergeSort(L); // even better: MergeSortPar(L)
    |
      create array R[1...n-m];
      for i from 1 to n-m do { R[i] := A[m+i]; }
      MergeSort(R); // even better: MergeSortPar(R)
    };

    Merge(L,R,A);   // desired: MergePRAM(L,R,A)
  }
}
```

Parallel MergeSort

Idea:

- parallelise "divide-and-conquer":
  recursive calls can be done in parallel
- use $p/2$ processors for each of the recursive calls
  (if p processors are available)

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Merging in Parallel?

- can Merge be executed in parallel?
- by how many processors?

Can Merge be Parallelised?

```
Merge (L:Array[1..p], R:Array[1..q], A:Array[1..n]) {
// merge the sorted arrays L and R into A (sorted)
// we presume that n=p+q
    i:=1; j:=1:
    for k from 1 to n do {
        if i > p
           then { A[k]:=R[j]; j=j+1; }
        else if j > q
           then { A[k]:=L[i]; i:=i+1; }
        else if L[i] < R[j]
           then { A[k]:=L[i]; i:=i+1; }
           else { A[k]:=R[j]; j:=j+1; }
    }
}
```

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    }
}
```

Problem: inherently sequential progress through arrays A, L, R

Odd-Even Merge

Ideas:

- start with a two sorted lists of length $n/2$:

2	3	4	7	1	5	6	8

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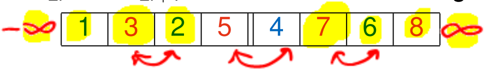
Observations

- final sequence is nearly sorted (only pairwise exchange required)
- odd- and even-indexed elements can be processed in parallel

Correctness of the Final Exchange Step

Claim (after odd/even sort):

- exchanges of a_{2i} and a_{2i+1} are sufficient for sorting



Proof:

- let O and E be sorted odd and even sequence, respectively; let A be sorted sequence
- add $E_0 = -\infty$ and $O_{n/2+1} = \infty$.
- for $i \in 0, \ldots, n/2$

$$A_{2i} = \min\{E_i, O_{i+1}\}$$
$$A_{2i+1} = \max\{E_i, O_{i+1}\}$$

note that A contains elements $A_0 = -\infty$ and $A_{n+1} = \infty$.

Correctness of the Final Exchange Step

$i = 0$ the first two elements in A are clearly $A_0 = -\infty$ and $A_1 = O_1$;

$i \geq 1$ using the induction hypothesis for $i' = 0, \ldots, i-1$ gives that the positions $A_0, \ldots, A_{2i-1}$ are composed from i even and i odd elements; hence, the next element is

$$A_{2i} = \min\{E_i, O_{i+1}\}$$

(note that E is indexed starting from 0 and O starting from 1)

now, we either have more odd or more even elements; however the number of even/odd elements within a prefix of A can at most differ by 1; therefore if the last element was odd we now have to choose the smallest even element (and vice versa); this gives

$$A_{2i+1} = \max\{E_i, O_{i+1}\}$$

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Counting Argument: x an odd-indexed element: $x = a_{2i+1}$

$6 = a_7$

- exactly i odd-indexed elements are smaller than x (sorted lists) $i = 3$
- $d_l, d_r =$ number of odd-indexed elements $< x$ in left/right half
  $\Rightarrow i = d_l + d_r$ $d_l = 2$ $d_r = 1$
- $v_l, v_r =$ number of even-indexed elements $< x$ in left/right half $v_l = 1$ $v_r = 1$
- x in left half: $v_l = d_l$, $v_r \in \{d_r, d_r - 1\}$
- x in right half: $v_l \in \{d_l, d_l - 1\}$, $v_r = d_r$
- **consequence:** $v_l + v_r \in \{d_l + d_r, d_l + d_r - 1\} = \{i, i - 1\}$

Correctness of the Final Exchange Step (2)

Counting Argument:

- count even- and odd-indexed elements $< x$ in both halves
- $v_l + v_r \in \{d_l + d_r, d_l + d_r - 1\} = \{i, i-1\}$

Possible Scenarios:

- $v_l + v_r = i \Rightarrow$ exactly i even elements $< x$
  $\Rightarrow$ i-th even-indexed element $a_{2i} < x \rightarrow$ **OK**
- $v_l + v_r = i - 1 \Rightarrow$ exactly $i - 1$ even elements $< x$
  therefore: $a_{2(i-1)} < x$, but $a_{2i} > x \rightarrow$ **exchange**
- in both cases:
  $a_{2(i+1)} > x$ (at most i even elements $< x$) $\rightarrow$ **OK**
  $a_{2(i-1)} < x$ (at least $i - 1$ even elements $< x$) $\rightarrow$ **OK**

$\Rightarrow$ only the left even-indexed neighbour of x can be out of place

OddEvenMerge – A First Try

```
OddEvenMerge_1 (A:Array[1..n]) {
// merge the sorted arrays A[1..n/2] and A[n/2+1..n]
// into A (sorted); n is a power of 2

    OddEvenSplit(A,Odd,Even);

    Sort(Odd);  Sort(Even);

    OddEvenJoin(A,Odd,Even);

    for i from 1 to n/2-1 do {
        if A[2i] > A[2i+1]
        then exchange A[2i] and A[2i+1]
    }
}
```

OddEvenSplit and OddEvenJoin (in parallel!)

```
OddEvenSplit (A:Array[1..n],
              Odd:Array[1..n/2], Even:Array[1..n/2]) {
   for i from 1 to n/2 do in parallel {
      Odd[i]  := A[2i-1];
      Even[i] := A[2i];
   }
}

OddEvenJoin (A:Array[1..n],
             Odd:Array[1..n/2], Even:Array[1..n/2]) {
   for i from 1 to n/2 do in parallel {
      A[2i-1] := Odd[i]   ;
      A[2i]   := Even[i];
   }
}
```

Towards a Better Implementation of OddEvenMerge

After OddEvenSplit:

- Odd consists of two halves that are already sorted
- Even consists of two halves that are already sorted

$\Rightarrow$ Odd and Even can be sorted using OddEvenMerge

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OddEvenMerge in Parallel:

- OddEvenSplit and OddEvenJoin are already parallel
- calls to OddEvenMerge can be executed in parallel (recursive calls will again issue parallel calls)
- final exchange loop can be parallelised

Parallel OddEvenMerge

```
OddEvenMergePRAM (A: Array [1..n]) {
   ! add stopping criterion:
   if n<=2 then { SortTwo(A); return; };

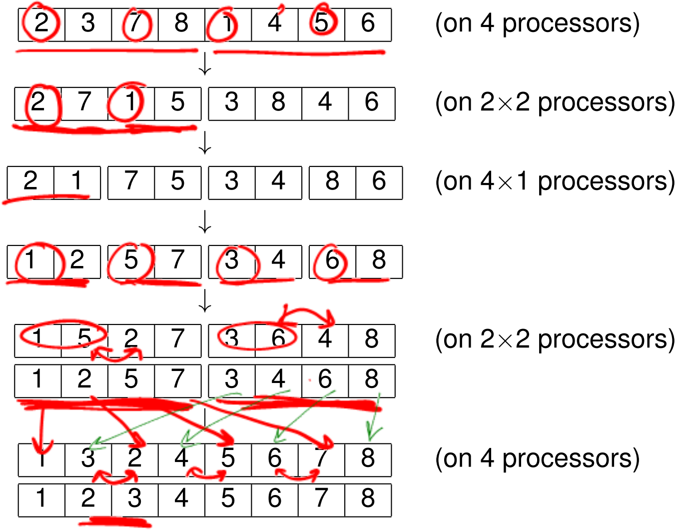
  OddEvenSplit(A, Odd, Even);

  do in parallel{ OddEvenMergePRAM(Odd);
                  OddEvenMergePRAM(Even); }

  OddEvenJoin(A, Odd, Even);

  for i from 1 to n/2−1 do in parallel {
     if A[2i] > A[2i+1]
     then exchange A[2i] and A[2i+1]
  }
}
```

Parallelism in OddEvenMerge



OddEvenMergeSort (in Parallel)

```
OddEvenMergeSortPRAM(A:Array[1..n]) {
 ! EREW PRAM with n/2 processors
 ! n assumed to be 2^k
 if n >= 2 then {

     do in parallel {
       OddEvenMergeSortPRAM(A[1..n/2]);
     |
       OddEvenMergeSortPRAM(A[n/2+1..n]);
     };

     OddEvenMergePRAM(A);
 }
}
```

Complexity of Odd-Even MergeSort

Complexity of OddEvenMerge:

- $\Theta(\log n)$ subsequent steps
- each step executed on $\frac{n}{2}$ processors
- total work: $\Theta(n \log n)$

Complexity of Odd-Even MergeSort:

- requires executions of OddEvenMerge on subarrays of lengths $k = 2, 4, \ldots, n$
- each OddEvenMerge step requires $\Theta(\log k)$ steps
- number of subsequent steps:

$$\log 2 + \log 4 + \cdots + \log n = \Theta\left((\log n)^2\right)$$

- total work: $\Theta\left(n(\log n)^2\right)$